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Generalized Navier–Stokes equations with initial data in local Q-type spaces [☆]

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ABSTRACT

In this paper, we establish a link between Leray mollified solutions of the three-dimensional generalized Navier–Stokes equations and mild solutions for initial data in the adherence of the test functions for the norm of $Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)$. This result applies to the usual incompressible Navier–Stokes equations.

1. Introduction

This paper studies the relationship between Leray mollified solutions and mild solutions to the generalized Navier–Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + (-\triangle)^\beta u + (u\cdot\nabla)u - \nabla p = 0, & \text{in } \mathbb{R}^{1+3}_+;\\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^{1+3}_+;\\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

for $\beta\in(1/2,1]$, where $(-\triangle)^\beta$ is the fractional Laplacian with respect to x defined by

$$\widehat{(-\triangle)^\beta u}(t,\xi) = |\xi|^{2\beta}\widehat{u}(t,\xi)$$

through Fourier transform. Here u_0 is the initial data, u and p are the velocity of the fluid and its pressure.

Eqs. (1.1) have been studied intensively, see Cannone [2], Giga and Miyakawa [10], Kato [11], Koch and Tataru [12], Xiao [23], Lions [17], Wu [19–22], Wu [18], Yuan [24], Zhai [25] and Zhou [26]. Recently, inspired by Xiao's paper [23], Li and Zhai [16] considered the well-posedness and regularity of Eqs. (1.1) with initial data in some new critical spaces $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$. In that paper, we proved that for initial data $u_0\in Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$ there exists a unique mild solution in the space $X^\beta_{\alpha;\infty}$, where the space $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)$ occurring above is a class of spaces which own a structure similar to the space $BMO^{-1}(\mathbb{R}^n)$ in [12] and $Q^{-1}_{\alpha;\infty}(\mathbb{R}^n)$ in [23]. It is easy to see that if $\alpha=-\frac{n}{2}$ and $\beta=1$, $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n)=BMO^{-1}(\mathbb{R}^n)$, and if $\alpha\in(0,1)$ and $\beta=1$,

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$Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n) = Q^{-1}_{\alpha;\infty}(\mathbb{R}^n)$. Here $Q^{-1}_{\alpha;\infty}(\mathbb{R}^n)$ is the derivative space of $Q_\alpha(\mathbb{R}^n)$, see Xiao [23], Dafni and Xiao [7,8], Essen, Janson, Peng and Xiao [9].

When $\beta = 1$, Eqs. (1.1) become the usual incompressible Navier–Stokes equations. In three-dimensional, the global existence, uniqueness and regularity of the solutions for the usual Navier–Stokes equations are long-standing open problem of fluid dynamics and the regularity problems is of course a millennium prize problem. Generally speaking, there are two specific approaches in the study of the existence of solutions to the three-dimensional incompressible Navier–Stokes equations. The first one is due to Leray [15] and the second is due to Kato [11]. We refer the readers to Bourgain and Pavlović [1], Cannone [3], Chen, Miao and Zhang [4–6], Lemarié-Rieusset [13], Zhou [27,28] and Zhou and Gala [29–31] for further information.

For general β, we can also define the mild and mollified solutions separately as follows. A mild solution to the generalized Navier–Stokes equations (1.1) is the fixed point of

$$A(u)(t,x) = e^{-t(-\Delta)^\beta}u_0(x) - B(u,u)(t,x), \tag{1.2}$$

where the bilinear form $B(u,v)$ is defined by

$$B(u,v) = \int_0^t e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla\cdot(u\otimes v)(s,x)\,ds.$$

Here $e^{-t(-\Delta)^\beta}$ denotes the convolution operator generated by the symbol

$$\big[\widehat{e^{-t(-\Delta)^\beta}}\big](\xi) = e^{-t|\xi|^{2\beta}}$$

and $\mathbb{P}$ denotes the Leray projector onto the divergence free vector filed. Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ with $\omega > 0$ and $\int_{\mathbb{R}^3}\omega(x)\,dx = 1$. Then for $\varepsilon > 0$, the mollified generalized Navier–Stokes equations are given by:

$$\begin{cases} \partial_t u + (-\triangle)^\beta u + \big((u*\omega_\varepsilon)\cdot\nabla\big)u - \nabla p = 0, & \text{in } \mathbb{R}^{1+3}_+;\\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^{1+3}_+;\\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^3 \end{cases} \tag{1.3}$$

with $\omega_\varepsilon(x) = \frac{1}{\varepsilon^3}\omega(\frac{x}{\varepsilon})$. The mollified solution to Eqs. (1.1) is the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of the solutions to (1.3), that is the fixed points for

$$T(u_\varepsilon)(t,x) = e^{-t(-\Delta)^\beta}u_0(x) - B_\varepsilon(u_\varepsilon,u_\varepsilon)(t,x), \tag{1.4}$$

with

$$B_\varepsilon(u,v) = \int_0^t e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla\cdot\big((u*\omega_\varepsilon)\otimes v\big)(s)\,ds.$$

The main goal of this paper is to establish a relation between the mild solutions obtained in [23] and [16] and the mollified solutions for Eqs. (1.3). In fact, our main results mean that for initial data $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)}$, with $\beta \in (1/2, 1]$, there exists $T > 0$ such that the sequence $\{u_\varepsilon\}_{\varepsilon>0} \in \overline{\mathcal{D}((0,T]\times\mathbb{R}^3)}^{X^\beta_{\alpha;T}(\mathbb{R}^3)}$ of solutions to (1.3) converges, when $\varepsilon \to 0$, to the mild solution obtained in [23] and [16], for $t \in (0,T)$. For the usual incompressible Navier–Stokes equations, when $\alpha = 0$, our main result goes back to Lemarié-Rieusset and Prioux's [14, Theorem 1.1]. However, it is worth pointing out that their theorem does not deduce our results even though $Q^{1,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)$ is a subspace of $bmo^{-1}(\mathbb{R}^3)$, since $X^1_{\alpha;T}(\mathbb{R}^3)$ is a proper subspace of $X^1_{0;T}(\mathbb{R}^3)$ when $0 < \alpha < 1$.

In the following, we give some definitions and known results. The first one is the space $Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^n)$ defined as follows.

Definition 1.1. For $\alpha > 0$ and $\max\{1/2,\alpha\} < \beta \leqslant 1$ with $\alpha + \beta - 1 \geqslant 0$, $Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^n)$ is the space of tempered distributions f on $\mathbb{R}^n$ such that, for all $T \in (0,\infty)$,

$$\sup_{0<r^{2\beta}<T}\sup_{x_0\in\mathbb{R}^n} r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-x_0|<r}\big|e^{-t(-\Delta)^\beta}f(y)\big|^2\frac{dy\,dt}{t^{\alpha/\beta}} < \infty.$$

The norm on $Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^n)} = \left(\sup_{0<r^{2\beta}<1}\sup_{x_0\in\mathbb{R}^n} r^{2\alpha-n+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-x_0|<r}\big|e^{-t(-\Delta)^\beta}f(y)\big|^2\frac{dy\,dt}{t^{\alpha/\beta}}\right)^{1/2}.$$

In [16], it was proved that the space $Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^n)$ consist of the derivatives of functions in $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ which is composed of all measurable functions with

$$\sup_I \big[l(I)\big]^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}}\,dx\,dy < \infty$$

where the supremum is taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

We now introduce the space $X^{\beta}_{\alpha;T}(\mathbb{R}^n)$.

Definition 1.2. Let $\alpha > 0$ and $\max\{1/2, \alpha\} < \beta \leqslant 1$ with $\alpha + \beta - 1 \geqslant 0$.

(i) A tempered distribution f on $\mathbb{R}^n$ belongs to $Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n)$ provided

$$\|f\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r^{2\beta}\in(0,T)} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} \big|K^{\beta}_t * f(y)\big|^2 t^{-\frac{\alpha}{\beta}}\,dy\,dt \right)^{1/2} < \infty.$$

(ii) A tempered distribution f on $\mathbb{R}^n$ belongs to $\overline{VQ^{\beta,-1}_{\alpha}}(\mathbb{R}^n)$ provided $\lim_{T\to 0}\|f\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n)} = 0$.

(iii) A function g on $\mathbb{R}^{1+n}_+$ belongs to the space $X^{\beta}_{\alpha;T}(\mathbb{R}^n)$ provided

$$\|g\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^n)} = \sup_{t\in(0,T)} t^{1-\frac{1}{2\beta}} \big\|g(t,\cdot)\big\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n,\, r^{2\beta}\in(0,T)} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} \big|g(t,y)\big|^2 t^{-\alpha/\beta}\,dy\,dt \right)^{1/2} < \infty.$$

In Xiao [23] and Li and Zhai [16], they proved the following well-posedness results about Eqs. (1.1) for $\beta = 1$ and $\frac{1}{2} < \beta < 1$, respectively.

Theorem 1.3. *Let $n \geqslant 2$, $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta \leqslant 1$ with $\alpha + \beta - 1 \geqslant 0$. Then:*

(i) *The generalized Navier–Stokes system* (1.1) *has a unique small global mild solution in $(X^{\beta}_{\alpha;\infty}(\mathbb{R}^n))^n$ for all initial data a with $\nabla \cdot a = 0$ and $\|a\|_{(Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n))^n}$ being small.*

(ii) *For any $T \in (0,\infty)$ there is an $\varepsilon > 0$ such that the generalized Navier–Stokes system* (1.1) *has a unique small mild solution in $(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n$ on $(0,T)\times\mathbb{R}^n$ when the initial data a satisfies $\nabla\cdot a = 0$ and $\|a\|_{(Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^n))^n} \leqslant \varepsilon$. In particular for all $a \in (\overline{VQ^{\beta,-1}_{\alpha}}(\mathbb{R}^n))^n$ with $\nabla\cdot a = 0$ there exists a unique small local mild solution in $(X^{\beta}_{\alpha;T}(\mathbb{R}^n))^n$ on $(0,T)\times\mathbb{R}^n$.*

Remark 1.4. The core of the proof of the above theorem is the following inequality: for u and $v \in X^{\beta}_{\alpha;T}(\mathbb{R}^n)$, we have

$$\big\|B(u,v)\big\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^n)} \leqslant \|u\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^n)} \|v\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^n)}. \tag{1.5}$$

The above inequality will play an important role in this paper.

We recall the definition of the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$ for $1 < p < \infty$.

Definition 1.5. Let $1 < p < \infty$. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$ if and only if

$$\forall\lambda > 0, \quad \big|\big\{x\in\mathbb{R}^n,\ \big|f(x)\big| > \lambda\big\}\big| \leqslant \frac{C}{\lambda^p}.$$

For $p = \infty$, we have $L^{\infty,\infty}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

In [14], the authors introduced a new class of Lorentz spaces.

Definition 1.6. For $1 < p < \infty$, the weak Lorentz space $\widetilde{L}^{p,\infty}((0,T))$ is the adherence of functions in $L^\infty((0,T))$ for the norm in the Lorentz space $L^{p,\infty}((0,T))$, that is,

$$\widetilde{L}^{p,\infty}((0,T)) = \overline{L^\infty((0,T))}^{L^{p,\infty}((0,T))}.$$

Let $\widetilde{L}^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$ be the space of all measurable functions f on $(0,T)\times\mathbb{R}^3$ such that $\|f(t,\cdot)\|_{L^{q,\infty}(\mathbb{R}^3)}\in\widetilde{L}^{p,\infty}((0,T))$. Then the following proposition holds.

Proposition 1.7. *(See [14, Proposition 2.7].) Let $T>0$, $1<p<\infty$ and $1<q\leqslant\infty$. The following properties are equivalent*:

(1) $f\in\widetilde{L}^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$.
(2) *For all $\lambda>0$, there exists a constant $C(\lambda)$ such that $C(\lambda)\to 0$ as $\lambda\to\infty$ and*

$$\left|\left\{\left\|f(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)}>\lambda\right\}\right|<\frac{C(\lambda)}{\lambda^p}.$$

(3) *For all $\varepsilon>0$, there exist $f_1\in L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))$ and $f_2\in L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$ such that*

$$\|f_2\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\leqslant\varepsilon\quad\text{and}\quad f=f_1+f_2.$$

The rest of this paper is organized as follows: In Section 2, we give two technical lemmas: Lemma 2.1—the continuity of the bilinear operator $B(\cdot,\cdot)$ in Lorentz spaces; Lemma 2.1—a local existence of mild solution to Eqs. (1.1) with initial data in Lorentz spaces. In Section 3, we establish main results of this paper. We only provide a proof for three spatial dimensions, but our proof goes through almost verbatim in higher dimensions.

2. Technical lemmas

In this section, we prove two preliminary lemmas. The first one can be regarded as a generalization of [14, Lemma 6.1] for the case $\beta=1$.

Lemma 2.1. *Let $T>0$ and $\frac{1}{2}<\beta<1$,*

$$B(u,v)(t,x)=\int_0^t e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla(u\otimes v)(s,x)\,ds.$$

Let $\frac{2\beta}{2\beta-1}\leqslant p<\infty$ and $\frac{3}{2\beta-1}<q\leqslant\infty$ such that $\beta-\frac{1}{2}=\frac{\beta}{p}+\frac{3}{2q}$. Then we have:

(1) *for $p\neq\frac{2\beta}{2\beta-1}$ (and so $q\neq\infty$),*

$$B:L^{p,\infty}\big((0,T),L^{q,\infty}\big(\mathbb{R}^3\big)\big)\times L^{p,\infty}\big((0,T),L^{q,\infty}\big(\mathbb{R}^3\big)\big)\to L^{p,\infty}\big((0,T),L^{q,\infty}\big(\mathbb{R}^3\big)\big)$$

with

$$\left\|B(u,v)\right\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\leqslant C\|u\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))};$$

(2) $B:L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))\times L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))\to L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))$ *with*

$$\left\|B(u,v)\right\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\leqslant C\|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))};$$

(3) $B:L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))\times L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))\to L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))$ *with*

$$\left\|B(u,v)\right\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\leqslant CT^{1/p}\|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))};$$

(4) $B:L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))\times L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))\to L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$ *with*

$$\left\|B(u,v)\right\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\leqslant CT^{1/p}\|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.$$

Proof. The proof of this lemma bases on the following inequality:

$$\left\|B(u,v)(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\leqslant\int_0^t(t-s)^{\frac{1}{2\beta}(-1-\frac{3}{q})}\left\|u(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds. \tag{2.1}$$

In fact, since

$$B(u,v)(t,x)=\int_0^t e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla(u\otimes v)(s,x)\,ds,$$

we have

$$\left\|B(u,v)(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \int_0^t \left\|e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla(u\otimes v)(s,\cdot)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds.$$

Since $e^{-u(-\Delta)^\beta}\mathbb{P}\nabla$ is a convolution operator, the Young inequality tells us for $1+\frac{1}{q}=\frac{2}{q}+\frac{1}{r}$,

$$\begin{aligned}\left\|B(u,v)(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)} &\leqslant \int_0^t \left\|e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla\right\|_{L^{r,\infty}(\mathbb{R}^3)}\left\|u\otimes v(s)\right\|_{L^{q/2,\infty}(\mathbb{R}^3)}\,ds\\ &\leqslant \int_0^t \left\|e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla\right\|_{L^{r,\infty}(\mathbb{R}^3)}\left\|u(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds.\end{aligned}$$

For $n=3$, the derivation of the generalized Oseen kernel satisfies

$$\left|e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla(x,y)\right| \leqslant \frac{1}{((t-s)^{\frac{1}{2\beta}}+|x-y|)^4}$$

(see [16, Lemma 4.10]). Then we can get

$$\begin{aligned}\left\|e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla\right\|_{L^{r,\infty}(\mathbb{R}^3)} &\leqslant \left\|e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla\right\|_{L^{r,\infty}(\mathbb{R}^3)}\\ &\leqslant \left(\int_{\mathbb{R}^3}\frac{1}{((t-s)^{\frac{1}{2\beta}}+|x-y|)^{4r}}\,dy\right)^{1/r}\\ &= \left(\int_0^\infty (t-s)^{-\frac{4r}{2\beta}}\frac{(|x-y|/t^{\frac{1}{2\beta}})^2}{(1+t^{-\frac{1}{2\beta}}|x-y|)^{4r}}t^{\frac{3}{2\beta}}\,d\big(t^{-\frac{1}{2\beta}}|x-y|\big)\right)^{1/r}\\ &= (t-s)^{\frac{3}{2\beta r}-\frac{4}{2\beta}}\left(\int_0^\infty \frac{\lambda^2}{(1+\lambda)^{4r}}\,d\lambda\right)^{1/r}\\ &\leqslant C(t-s)^{\frac{1}{2\beta}(\frac{3}{r}-4)}.\end{aligned}$$

For $1=\frac{1}{r}+\frac{1}{q}$, we have

$$\begin{aligned}\left\|B(u,v)(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)} &\leqslant \int_0^t (t-s)^{\frac{1}{2\beta}(\frac{3}{r}-4)}\left\|u(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds\\ &= \int_0^t (t-s)^{\frac{1}{2\beta}(-1-\frac{3}{q})}\left\|u(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds.\end{aligned}$$

This completes the proof of (2.1). Now we prove (1)–(4) by using (2.1).

(1) Since $\beta-\frac{1}{2}=\frac{\beta}{p}+\frac{3}{2q}$, we know $1+\frac{1}{p}=\frac{1}{p}+\frac{1}{p}+\frac{1}{r_1}$ with $\frac{1}{r_1}=\frac{1}{2\beta}+\frac{3}{2\beta q}$. By Young's inequality, we get

$$\left\|B(u,v)\right\|_{L^p((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant \left\|s^{-\frac{1}{2\beta}-\frac{3}{2\beta q}}\right\|_{L^{r_1,\infty}}\|u\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.$$

Now we compute the norm $\|s^{-\frac{1}{r_1}}\|_{L^{r_1,\infty}(0,T)}$, where

$$\|f\|_{L^{r_1,\infty}(0,T)} = \sup_\lambda \lambda\left|\left\{t\in(0,T),\ \left|f(t)\right|>\lambda\right\}\right|^{1/p}.$$

If $s\in(0,T)$ and $T<\frac{1}{\lambda^{r_1}}$, then

$$\lambda\left|\left\{t\in(0,T),\ s^{-\frac{1}{r_1}}>\lambda\right\}\right|^{1/r_1} \leqslant \lambda T^{1/r_1} \leqslant 1.$$

If $T > \frac{1}{\lambda^{r_1}}$, then $T^{-\frac{1}{r_1}} < \lambda$. For $s^{-\frac{1}{r_1}} > \lambda$, we can find an s_0 such that $s_0 = \frac{1}{\lambda^{r_1}}$. When $0 < s < s_0$, $s^{-\frac{1}{r_1}} > s_0^{-\frac{1}{r_1}} = \lambda$, then we have

$$\lambda\left|\left\{s \in (0,T)\colon s^{-\frac{1}{r_1}} > \lambda\right\}\right|^{1/r_1} \leqslant \lambda s_0^{1/r_1} = 1.$$

Therefore we get $s^{-\frac{1}{r_1}} \in L^{r_1,\infty}((0,T))$ and $\|s^{-\frac{1}{r_1}}\|_{L^{r_1,\infty}(0,T)} \leqslant 1$.

(2) It follows from (2.1) that

$$\left\|B(u,v)(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \int_0^t (t-s)^{-\frac{1}{2\beta}-\frac{3}{2\beta q}} \left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds.$$

Then we obtain

$$\left\|B(u,v)(t)\right\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant \|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \sup_{0<t<T}\left|\int_0^t (t-s)^{-\frac{1}{2\beta}-\frac{3}{2\beta q}} \left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds\right|.$$

By Hölder's inequality with $1 = \frac{1}{p} + \frac{1}{2\beta} + \frac{3}{2\beta q}$, we have

$$\left\|B(u,v)(t)\right\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant C\|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.$$

(3) By (2.1), we get

$$\begin{aligned}\left\|B(u,v)\right\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} &\leqslant \|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \int_0^t (t-s)^{-(\frac{1}{2\beta}+\frac{3}{2\beta q})}\,ds \\ &\leqslant CT^{1/p}\|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}.\end{aligned}$$

(4) (2.1) and Young's inequality with $1 + \frac{1}{p} = \frac{1}{p} + \frac{1}{p} + (\frac{1}{2\beta} + \frac{3}{2\beta q})$ imply that

$$\begin{aligned}\left\|B(u,v)(t)\right\|_{L^{q,\infty}(\mathbb{R}^3)} &\leqslant \int_0^t (t-s)^{-(\frac{1}{2\beta}+\frac{3}{2\beta q})}\left\|u(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds \\ &\leqslant \|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\left(\int_0^t (t-s)^{-(\frac{1}{2\beta}+\frac{3}{2\beta q})}\left\|v(s)\right\|_{L^{q,\infty}(\mathbb{R}^3)}\,ds\right).\end{aligned}$$

Since $\|f * g\|_{p,\infty} \leqslant \|f\|_{p,\infty}\|g\|_1$, we have

$$\begin{aligned}\left\|B(u,v)\right\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} &\leqslant \|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \sup_{0<t<T}\int_0^t s^{-1+\frac{1}{p}}\,ds \\ &\leqslant CT^{1/p}\|u\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.\end{aligned}$$

This completes the proof of Lemma 2.1. □

We need the following local existence of solution to Eqs. (1.1) with initial data in Lorentz spaces.

Lemma 2.2. *Let $\frac{1}{2} < \beta \leqslant 1$, $\frac{3}{2\beta-1} < q \leqslant \infty$, $\frac{2\beta}{2\beta-1} < p \leqslant \infty$ and $u_0 \in L^{q,\infty}(\mathbb{R}^3)$. For $T > 0$ such that $4T^{1/p}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} < 1$, there exists a mild solution $u \in L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))$ to Eqs. (1.1), which is unique in the ball centered at 0, of radius $2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}$.*

Proof. We construct $\{e_n\}$ as follows:

$$\begin{cases} e_{n+1} = e_0 - B(e_n,e_n), \\ e_0 = e^{-t(-\Delta)^\beta}u_0. \end{cases} \tag{2.2}$$

We claim that $\|e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}$. For $n=0$, by Young's inequality, we have

$$\begin{aligned}\|e_0\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} &= \big\|e^{-t(-\Delta)^\beta}u_0\big\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} = \sup_{t\in(0,T)}\big\|e^{-t(-\Delta)^\beta}u_0\big\|_{L^{q,\infty}(\mathbb{R}^3)}\\ &\leqslant \big\|e^{-t(-\Delta)^\beta}\big\|_{L^1(\mathbb{R}^3)}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \bigg(\int_{\mathbb{R}^3}\frac{1}{t^{\frac{3}{2\beta}}}\frac{1}{(1+\frac{|x-y|}{t^{1/2\beta}})^3}\,dy\bigg)\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}\\ &\leqslant 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}.\end{aligned}$$

Assume that the estimate is true for some $n\in\mathbb{N}$. For $n+1$, we get

$$\begin{aligned}\|e_{n+1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} &\leqslant \|e_0\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + \big\|B(e_n,e_n)\big\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\\ &\leqslant \|e_0\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + T^{1/p}\|e_n\|^2_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\\ &\leqslant \|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} + 4T^{1/p}\|u_0\|^2_{L^{q,\infty}(\mathbb{R}^3)}\\ &\leqslant 2\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}.\end{aligned}$$

This tells us

$$\begin{aligned}\|e_{n+1}-e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} &= \big\|B(e_n,e_n)-B(e_{n-1},e_{n-1})\big\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\\ &\leqslant T^{1/p}\|e_n-e_{n-1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\big(\|e_n\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}+\|e_{n-1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}\big)\\ &\leqslant 4T^{1/p}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)}\|e_n-e_{n-1}\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))}.\end{aligned}$$

Since $4T^{1/p}\|u_0\|_{L^{q,\infty}(\mathbb{R}^3)} < 1$, the Picard contraction principle guarantees this lemma. □

3. Main results

In this section, we state and prove our main results. First, we need the following proposition which generalizes the case $\beta=1$ established by Lemarié-Rieusset and Prioux [14].

Proposition 3.1. *Let $T>0$, $\frac{1}{2}<\beta<1$ and u, v be two mild solutions to Eqs.* (1.1) *belonging to the space $\widetilde{L}^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))$ with $\frac{2\beta}{2\beta-1}<p<\infty$ and $\frac{3}{2\beta-1}<q<\infty$ such that $\beta-\frac{1}{2}=\frac{\beta}{p}+\frac{3}{2q}$. Assume that there exists $\theta\in(0,T)$ such that $u(\theta)=v(\theta)$. Then u and v are equal for $t\in(\theta,T]$.*

Proof. Let $t_0>0$ and $\lambda>0$. We can split u and v into:

$$u=u_\lambda+u'_\lambda \quad\text{and}\quad v=v_\lambda+v'_\lambda$$

where $u_\lambda = u\chi_{\{t:\ \|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)}>\lambda\}}$ and $v_\lambda = v\chi_{\{t:\ \|v(t)\|_{q,\infty(\mathbb{R}^3)}>\lambda\}}$. By construction and the definition of the Lorentz spaces (see Proposition 1.7) we have

$$\big\|u'_\lambda\big\|_{L^\infty((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant\lambda, \qquad \|u_\lambda\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant C(\lambda)$$

and the same estimates hold true for v'_λ and v_λ. Then, we compute by Lemma 2.1(1),

$$\begin{aligned}&\|u-v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\\ &\quad\leqslant \big\|B(u,u)-B(v,v)\big\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\\ &\quad\leqslant C_0\big\|B(u-v,u)\big\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))} + C_0\big\|B(v,u-v)\big\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\\ &\quad\leqslant C\|u-v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\big(\|u\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}+\|v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\big).\end{aligned}$$

Since $u=u_\lambda+u'_\lambda$ with $\|u_\lambda\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant C(\lambda)$ and $\|u'_\lambda\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant\lambda$, we get $\|u\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant C(\lambda)+\lambda t_0^{1/p}$. The same estimate holds for v. Hence we can obtain

$$\|u-v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant C_0\big(2C(\lambda)+2\lambda t_0^{1/p}\big)\|u-v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}.$$

We choose $\lambda>0$ large enough to guarantee $2C_0C(\lambda)<1/4$ and choose $t_0>0$ small enough such that $C_0t_0^{1/p}<1/4$. Thus there exists $\delta<1$ satisfies

$$\|u-v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}\leqslant\delta\|u-v\|_{L^{p,\infty}((\theta,\theta+t_0),L^{q,\infty}(\mathbb{R}^3))}.$$

So $u=v$ for $t\in(\theta,\theta+t_0)$. For T, there exists n such that $T<\theta+nt_0$. Thus $u=v$ for $t\in(\theta,T]$. □

Lemma 3.2. *(See [14, Proposition 2.9].) Let $T > 0$ and $1 \leqslant p, q \leqslant \infty$. If u satisfies*

$$\begin{cases} \sup\limits_{t\in(0,T)} t^{1/p}\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} < \infty, \\ t^{1/p}\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \to 0 \quad (\text{as } t \to 0), \end{cases} \tag{3.1}$$

then the function u belongs to the space $\widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$.

To establish the equivalence between the mild and mollified solution to the (*GNS*) equations, we need the following lemma.

Lemma 3.3. *Let $v \in \mathcal{D}((0,T] \times \bar{B}(0,R))$ and $R > 0$ such that $\operatorname{supp} v \subset (0,T] \times \bar{B}(0,R)$ where supp denotes the supports of the function v and $\bar{B}(0,R)$ the closed ball of radius R centered at 0. Then, for $t \in (0,T]$ and $y \in \mathbb{R}^3$ such that $|y| \geqslant \lambda R$ for $\lambda > 1$, we have for some constant $C > 0$,*

$$\big|B(u,u)(t,y)\big| \leqslant \frac{C}{(\lambda R)^4}\|u\|^2_{L^2((0,T]\times\mathbb{R}^3)}.$$

Proof. Since $|y| \geqslant \lambda R > \lambda|z|$, $|y-z| \geqslant (1-\frac{1}{\lambda})|y|$. Then, we can get

$$\begin{aligned} \big|B(u,u)(t,y)\big| &= \left|\int_0^t e^{-(t-s)(-\Delta)^\beta}\mathbb{P}\nabla(u\otimes u)\,ds\right| \leqslant C\int_0^t\int_{\mathbb{R}^3}\frac{1}{((t-s)^{\frac{1}{2\beta}}+|z-y|)^4}\big|v(s,z)\big|^2\,ds\,dz \\ &\leqslant C\int_0^t\int_{\mathbb{R}^3}\frac{1}{|z-y|^4}\big|v(s,z)\big|^2\,ds\,dz \leqslant C\int_0^t\int_{B(0,R)}\frac{1}{|z-y|^4}\big|v(s,z)\big|^2\,ds\,dz \\ &\leqslant C\frac{1}{|y|^4}\int_0^t\int_{B(0,R)}\big|v(s,z)\big|^2\,ds\,dz = C\frac{1}{|y|^4}\|v\|^2_{L^2((0,T)\times\mathbb{R}^3)} \lesssim \frac{1}{|\lambda R|^4}\|v\|^2_{L^2((0,T)\times\mathbb{R}^3)}. \qquad \square \end{aligned}$$

Theorem 3.4. *Let $\alpha > 0$, $\max\{\frac{1}{2},\alpha\} < \beta \leqslant 1$ with $\alpha+\beta-1 \geqslant 0$ and let $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)}$ such that $\nabla\cdot u_0 = 0$ and $T > 0$ small enough to ensure $\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} < \frac{1}{4C}$. Then there exists a mild solution $u \in \overline{\mathcal{D}((0,T)\times\mathbb{R}^3)}^{X^\beta_{\alpha;T}(\mathbb{R}^3)}$ to Eqs. (1.1).*

Proof. We construct $\{v_n\}_{n\in\mathbb{N}}$ by

$$\begin{cases} v_n = v_0 - B(v_{n-1}, v_{n-1}), \quad \text{for } n \geqslant 1, \\ v_0 = e^{-t(-\Delta)^\beta}u_0. \end{cases} \tag{3.2}$$

For $n = 0$. By assumption, if $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)}$, there exists a sequence $u_0^m \in \mathcal{D}(\mathbb{R}^3)$ such that $\|u_0 - u_0^m\|_{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)} \to 0$ as $m \to \infty$. From the definition of $Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)$, if $f \in Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)$,

$$\sup_{0<t^{2\beta}<T}\ \sup_{x_0\in\mathbb{R}^3} t^{2\alpha-3+2\beta-2}\int_0^{t^{2\beta}}\int_{|x-x_0|<t}\big|e^{-s(-\Delta)^\beta}f(x)\big|^2\frac{ds\,dx}{s^{\alpha/\beta}} < \infty.$$

Hence, as $m \to \infty$,

$$\sup_{0<t^{2\beta}<T}\ \sup_{x_0\in\mathbb{R}^3} t^{2\alpha-3+2\beta-2}\int_0^{t^{2\beta}}\int_{|x-x_0|<t}\big|e^{-s(-\Delta)^\beta}\big(u_0-u_0^m\big)(x)\big|^2\frac{ds\,dx}{s^{\alpha/\beta}} \to 0.$$

From the embedding: $Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^3) \hookrightarrow \dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^3)$ (see [16, Theorem 4.6]), we obtain

$$t^{\frac{2\beta-1}{2\beta}}\big\|e^{-t(-\Delta)^\beta}f\big\|_{L^\infty(\mathbb{R}^3)} \leqslant C\|f\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^3)}.$$

By the definition of $X^{\beta}_{\alpha;T}(\mathbb{R}^3)$, we get $\|e^{-t(-\Delta)^{\beta}} f\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} \leqslant C\|f\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^3)}$. Then we have

$$\left\|e^{-t(-\Delta)^{\beta}}\left(u_0-u_0^m\right)\right\|_{X^{\beta,-1}_{\alpha;T}(\mathbb{R}^3)} \leqslant C\left\|u_0-u_0^m\right\|_{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^3)}$$

and $\|e^{-t(-\Delta)^{\beta}}(u_0-u_0^m)\|_{X^{\beta,-1}_{\alpha;T}(\mathbb{R}^3)} \to 0$ as $m\to 0$. So $e^{-t(-\Delta)^{\beta}}u_0 = \lim_{m\to\infty} e^{-t(-\Delta)^{\beta}}u_0^m$ in $X^{\beta}_{\alpha;T}(\mathbb{R}^3)$. It follows from $e^{-t(-\Delta)^{\beta}}u_0^m \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ that

$$v_0 = e^{-t(-\Delta)^{\beta}}u_0 \in \overline{\mathcal{D}\left((0,T)\times\mathbb{R}^3\right)}^{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}.$$

Let us assume $v_{n-1} \in \overline{\mathcal{D}((0,T)\times\mathbb{R}^3)}^{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}$. For v_n, since $v_n = e^{-t(-\Delta)^{\beta}}u_0 - B(v_{n-1},v_{n-1})$,

$$u_0 \in \overline{\mathcal{D}\left(\mathbb{R}^3\right)}^{Q^{\beta,-1}_{\alpha;T}(\mathbb{R}^3)} \quad\Longrightarrow\quad v_0 = e^{-t(-\Delta)^{\beta}}u_0 \in \overline{\mathcal{D}\left((0,T)\times\mathbb{R}^3\right)}^{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}.$$

We only need to prove $B(v_{n-1},v_{n-1}) \in \overline{\mathcal{D}((0,T)\times\mathbb{R}^3)}^{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}$.

By induction, there exists a sequence $v^m_{n-1} \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ such that

$$\left\|v_{n-1}-v^m_{n-1}\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} \to 0, \quad \text{as } m\to\infty.$$

Since v^m_{n-1} is compact supported in time and space, we have $B(v^m_{n-1},v^m_{n-1}) \in C^{\infty}((0,T]\times\mathbb{R}^3)$ and is of compact support in time. Let $\{\varphi_m\}_{m\in\mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^3)$ such that for each $m\in\mathbb{N}$, $\|\varphi_m\|_\infty = 1$. Assume $\operatorname{supp}\varphi_m \subset \bar{B}(0,\lambda_m R_m+1)$ and $\varphi_m(x)=1$ if $x\in B(0,\lambda_m R_m)$ where $R_m>0$ is such that $\operatorname{supp} v^m_{n-1} \subset (0,T]\times B(0,R_m)$ and $\lambda_m > m\|v^m_{n-1}\|^{1/2}_{L^2((0,T)\times\mathbb{R}^3)}$. We denote $B^m(v_{n-1},v_{n-1}) = \varphi_m \times B(v^m_{n-1},v^m_{n-1})$ and get

$$\begin{aligned}
&\left\|B(v_{n-1},v_{n-1})-B^m\left(v^m_{n-1},v^m_{n-1}\right)\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}\\
&\quad\leqslant \left\|B(v_{n-1},v_{n-1})-B\left(v^m_{n-1},v^m_{n-1}\right)\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} + \left\|(1-\varphi_m)B\left(v^m_{n-1},v^m_{n-1}\right)\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}\\
&\quad\leqslant C\left\|v_{n-1}-v^m_{n-1}\right\|_{X^{\beta}_{\alpha;T}}\left[\|v_{n-1}\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} + \left\|v^m_{n-1}\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}\right] + \left\|(1-\varphi_m)B\left(v^m_{n-1},v^m_{n-1}\right)\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}.
\end{aligned}$$

Since φ_m is supported on $\bar{B}(0,\lambda_m R_m+1)$ and $\varphi_m=1$ on $B(0,\lambda_m R_m)$, $(1-\varphi_m(y))$ is supported on $\bar{B}^c(0,\lambda_m R_m)=\{y: |y|>\lambda_m R_m\}$. Then, we obtain

$$\begin{aligned}
&\left\|(1-\varphi_m)B\left(v^m_{n-1},v^m_{n-1}\right)\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}\\
&\quad\leqslant \sup_{t\in(0,T)} t^{1-\frac{1}{2\beta}}\left\|(1-\varphi_m)B\left(v^m_{n-1},v^m_{n-1}\right)\right\|_{L^\infty(\mathbb{R}^3)}\\
&\qquad + \sup_{t^{2\beta}\in(0,T)}\sup_{x_0\in\mathbb{R}^3}\left(t^{2\alpha-3+2\beta+2\beta-2}\int_0^{t^{2\beta}}\int_{|y-x_0|<t}\left|(1-\varphi_m)B\left(v^m_{n-1},v^m_{n-1}\right)(s,y)\right|^2\frac{ds\,dy}{s^{\alpha/\beta}}\right)^{1/2}\\
&\quad\leqslant \sup_{t\in(0,T)} t^{1-\frac{1}{2\beta}}\frac{1}{(\lambda_m R_m)^4}\left\|v^m_{n-1}\right\|^2_{L^2((0,T]\times\mathbb{R}^3)}\\
&\qquad + \sup_{t^{2\beta}\in(0,T)}\sup_{x_0\in\mathbb{R}^3}\left\|v^m_{n-1}\right\|^2_{L^2((0,T]\times\mathbb{R}^3)}\frac{1}{(\lambda_m R_m)^4}\left(t^{2\alpha-3+2\beta+2\beta-2}\int_0^{t^{2\beta}}\int_{|y-x_0|<t}\frac{ds\,dy}{s^{\alpha/\beta}}\right)^{1/2}\\
&\quad\leqslant CT^{1-\frac{1}{2\beta}}\frac{1}{(\lambda_m R_m)^4}\left\|v^m_{n-1}\right\|^2_{L^2((0,T]\times\mathbb{R}^3)} + \frac{\|v^m_{n-1}\|^2_{L^2((0,T]\times\mathbb{R}^3)}}{(\lambda_m R_m)^4}\sup_{t^{2\beta}\in(0,T)}\left(t^{2\alpha-3+2\beta-2}t^{3+2\beta(1-\alpha/\beta)}\right)^{1/2}\\
&\quad\leqslant CT^{1-\frac{1}{2\beta}}\frac{1}{(\lambda_m R_m)^4}\left\|v^m_{n-1}\right\|^2_{L^2((0,T]\times\mathbb{R}^3)} \leqslant CT^{1-\frac{1}{2\beta}}\frac{1}{(mR_m)^4} \to 0 \quad (\text{as } m\to\infty).
\end{aligned}$$

Thus, $\|B(v_{n-1},v_{n-1})-B^m(v_{n-1},v_{n-1})\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} \to 0$ as $m\to 0$, that is, $v_n \in \overline{\mathcal{D}((0,T]\times\mathbb{R}^3)}^{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}$.

Next we prove that v_n have a limit in $X^{\beta}_{\alpha;T}(\mathbb{R}^3)$. We prove $\|v_n\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} \leqslant 2\|e^{-t(-\Delta)^{\beta}}u_0\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}$. It follows from $v_0 = e^{-t(-\Delta)^{\beta}}u_0$ that

$$\|v_0\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} = \left\|e^{-t(-\Delta)^{\beta}}u_0\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)} \leqslant 2\left\|e^{-t(-\Delta)^{\beta}}u_0\right\|_{X^{\beta}_{\alpha;T}(\mathbb{R}^3)}.$$

We assume that for $n \in \mathbb{N}$, $\|v_n\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} \leqslant 2\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}$. Then we get

$$\begin{aligned}\|v_{n+1}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} &\leqslant \|v_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} + \|B(v_n,v_n)\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} \leqslant \|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} + C\|v_n\|^2_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\\ &\leqslant \|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} + 4C\|e^{-t(-\Delta)^\beta}u_0\|^2_{X^\beta_{\alpha;T}(\mathbb{R}^3)}.\end{aligned}$$

It follows from $\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} < \frac{1}{4C}$ that $\|v_{n+1}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} \leqslant 2\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}$. Moreover,

$$\begin{aligned}\|v_n - v_{n-1}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} &\leqslant \|B(v_{n-1},v_{n-1}) - B(v_{n-2},v_{n-2})\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\\ &\leqslant C\|v_{n-1}-v_{n-2}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\big(\|v_{n-1}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} + \|v_{n-2}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\big)\\ &\leqslant 4C\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|v_{n-1}-v_{n-2}\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\\ &\leqslant \big(4C\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\big)^n\|v_1-v_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}.\end{aligned}$$

Since $4C\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} < 1$, the Picard contraction principle implies the desired. □

Theorem 3.5. *Let $\alpha > 0$, $\max\{\frac{1}{2},\alpha\} < \beta \leqslant 1$ with $\alpha+\beta-1 \geqslant 0$ and let $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)}$ such that $\nabla\cdot u_0 = 0$ and $T > 0$ is small enough to ensure $\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} < \frac{1}{4C}$. Then for $\varepsilon > 0$, there exists a solution $u_\varepsilon \in \overline{\mathcal{D}((0,T]\times\mathbb{R}^3)}^{X^\beta_{\alpha;T}(\mathbb{R}^3)}$ to the mollified generalized Navier–Stokes equations* (1.3)*.*

Proof. We only need to prove $\|f*\omega_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} \leqslant \|f\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}$. In fact, we have $\|\omega_\varepsilon * f\|_{L^\infty(\mathbb{R}^3)} \leqslant \|\omega_\varepsilon\|_{L^1(\mathbb{R}^3)}\|f\|_{L^\infty(\mathbb{R}^3)}$ and

$$\begin{aligned}&\left(r^{2\alpha-3+2\beta-2}\int_0^{r^{2\beta}}\int_{|x-x_0|<r}\big|f*\omega_\varepsilon(t,x)\big|^2\frac{dt\,dx}{t^{\alpha/\beta}}\right)^{1/2}\\ &\quad\leqslant \left(r^{2\alpha-3+2\beta-2}\int_0^{r^{2\beta}}\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\chi_{B(x_0,r)}f(t,x-y)\omega_\varepsilon(y)\,dy\right|^2\frac{dt\,dx}{t^{\alpha/\beta}}\right)^{1/2}\\ &\quad\leqslant \int_{\mathbb{R}^3}\big|\omega_\varepsilon(y)\big|\left(r^{2\alpha-3+2\beta-2}\int_0^{r^{2\beta}}\int_{\mathbb{R}^3}\big|f(t,x-y)\big|^2\chi_{B(x_0,r)}\frac{dt\,dx}{t^{\alpha/\beta}}\right)^{1/2}dy\\ &\quad\leqslant \int_{\mathbb{R}^3}\big|\omega_\varepsilon(y)\big|\left(r^{2\alpha-3+2\beta-2}\int_0^{r^{2\beta}}\int_{|x_1-(x_0-y)|<r}\big|f(t,x_1)\big|^2\frac{dt\,dx_1}{t^{\alpha/\beta}}\right)^{1/2}dy\\ &\quad\leqslant \int_{\mathbb{R}^3}\big|\omega_\varepsilon(y)\big|\,dy\sup_{z\in\mathbb{R}^3}\sup_{r^{2\beta}\in(0,T]}\left(r^{2\alpha-3+2\beta-2}\int_0^{r^{2\beta}}\int_{|x_1-z|<r}\big|f(t,x_1)\big|^2\frac{dt\,dx_1}{t^{\alpha/\beta}}\right)^{1/2}\\ &\quad\leqslant \|\omega_\varepsilon\|_{L^1(\mathbb{R}^3)}\|f\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}.\end{aligned}$$

Similar to the proof of Theorem 3.4, we can complete the proof. □

Theorem 3.6. *For $\alpha > 0$, $\max\{\frac{1}{2},\alpha\} < \beta \leqslant 1$ with $\alpha+\beta-1 \geqslant 0$, let $u_0 \in \overline{\mathcal{D}(\mathbb{R}^3)}^{Q^{\beta,-1}_{\alpha,\mathrm{loc}}(\mathbb{R}^3)}$ and $T > 0$ be given in Theorem 3.4. Then the sequence of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to the mollified equations* (1.3) *obtained by Theorem 3.5 converges strongly, as ε tends to 0, to the mild solution u to Eqs.* (1.1) *obtained by Picard contraction principle, of Theorem 3.4.*

Proof. For the bilinear form $B(u,v)$, we have

$$\begin{aligned}u - u_\varepsilon &= B(u,u) - B_\varepsilon(u_\varepsilon,u_\varepsilon) = B(u,u) - B(u_\varepsilon*\omega_\varepsilon,u_\varepsilon)\\ &= B(u,u-u_\varepsilon) + B\big(u-(u*\omega_\varepsilon),u_\varepsilon\big) + B\big((u-u_\varepsilon)*\omega_\varepsilon,u_\varepsilon\big)\end{aligned}$$

and

$$\begin{aligned}\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} &\leqslant C\|u\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}+C\big\|u-(u*\omega_\varepsilon)\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\\ &\quad+C\big\|(u-u_\varepsilon)*\omega_\varepsilon\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)(\mathbb{R}^3)}\|u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\\ &:=A_1+A_2+A_3\end{aligned}$$

where $A_3\leqslant C\|\omega_\varepsilon\|_{L^1(\mathbb{R}^3)}\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}$. Hence we have

$$\begin{aligned}\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)} &\leqslant 2C\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}+2C\big\|u-(u*\omega_\varepsilon)\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\\ &\leqslant 4C\big\|e^{-t(-\Delta)^\beta}u_0\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}+2C\big\|e^{-t(-\Delta)^\beta}u_0\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\big\|u-(u*\omega_\varepsilon)\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}.\end{aligned}$$

This tells us

$$\|u-u_\varepsilon\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\leqslant\frac{2C\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}}{1-4C\|e^{-t(-\Delta)^\beta}u_0\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}}\big\|u-(u*\omega_\varepsilon)\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}.$$

Since $\omega_\varepsilon\in\mathcal{D}(\mathbb{R}^3)$, $\omega_\varepsilon*u\in\mathcal{D}(\mathbb{R}^3\times(0,T))$. Thus, for $u\in\overline{\mathcal{D}(\mathbb{R}^3\times(0,T))}^{X^\beta_{\alpha;T}(\mathbb{R}^3)}$,

$$\big\|u-(u*\omega_\varepsilon)\big\|_{X^\beta_{\alpha;T}(\mathbb{R}^3)}\to 0,\quad\text{as }\varepsilon\to 0.\qquad\Box$$

Now we recall a class of weak Besov spaces which can be found in [14].

Definition 3.7. Let $\alpha>0$, $1<q<\infty$. We denote by $\widetilde{B}_q^{-\alpha,\infty}(\mathbb{R}^3)$ the adherence of functions in $L^q(\mathbb{R}^3)$ for the norm of $B_q^{-\alpha,\infty}(\mathbb{R}^3)$ and by $\widetilde{B}_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)$ for functions in $L^{q,\infty}(\mathbb{R}^3)$ for the norm of $B_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)=B_{L^q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)$, that is,

$$\widetilde{B}_q^{-\alpha,\infty}\big(\mathbb{R}^3\big)=\overline{L^q\big(\mathbb{R}^3\big)}^{B_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)}\quad\text{and}\quad\widetilde{B}_{q,\infty}^{-\alpha,\infty}\big(\mathbb{R}^3\big)=\overline{L^{q,\infty}\big(\mathbb{R}^3\big)}^{B_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)}.$$

Lemma 3.8. *Let $\frac{1}{2}<\beta<1$ and let $\alpha>0$ and $1<q<\infty$. If $u\in\widetilde{B}_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)$, then*

$$\begin{cases}\sup\limits_{0<t<1} t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}<\infty,\\ t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}\to 0\quad(\text{as }t\to 0).\end{cases}\tag{3.3}$$

Proof. Since $u\in\widetilde{B}_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)$, we have $u\in B_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)$. Then

$$\sup_{t>0}t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}<\infty$$

and there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ of functions in $L^{q,\infty}(\mathbb{R}^3)$ such that

$$\big\|(u_n-u)(t)\big\|_{B_{q,\infty}^{-\alpha,\infty}(\mathbb{R}^3)}\to 0,\quad\text{as }t\to 0.$$

So there exists $N>0$ such that for $n>N$,

$$\sup_{t>0}t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}(u_n-u)(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}<\frac{\varepsilon}{2}.$$

Then for all $t>0$ we have, by Young's inequality,

$$t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}<\frac{\varepsilon}{2}+t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}u_{N+1}(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}<\frac{\varepsilon}{2}+Ct^{\alpha/2\beta}\big\|u_{N+1}(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}.$$

Let $t_0=\varepsilon^{2\beta/\alpha}(2C\|u_{N+1}(t)\|_{L^{q,\infty}(\mathbb{R}^3)})^{-(2\beta/\alpha)}$, we see that for $t<t_0$,

$$t^{\alpha/2\beta}\big\|e^{-t(-\Delta)^\beta}u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\qquad\Box$$

The following result gives us a condition for initial data under which the solution to Eqs. (1.1) for $\beta\in(1/2,1)$ belongs to the weak Lorentz spaces. Similar results hold for $\beta=1$, see Lemarié-Rieusset and Prioux [14].

Theorem 3.9. *Let $\frac{1}{2} < \beta < 1$ and let $\frac{2\beta}{2\beta-1} < p \leqslant \infty$ and $\frac{3}{2\beta-1} < q \leqslant \infty$ such that $\frac{\beta}{p} + \frac{3}{2q} = \beta - \frac{1}{2}$ and $u_0 \in \widetilde{B}^{-\alpha,\infty}_{L^{q,\infty}}(\mathbb{R}^3)$ such that $\nabla \cdot u_0$. Then there exist $T > 0$ and a mild solution u to Eqs. (1.1) in the space $\widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$.*

Proof. We construct the sequence $\{v_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} v_n = v_0 - B(v_{n-1}, v_{n-1}) \quad \text{for } n \geqslant 1, \\ v_0 = e^{-t(-\Delta)^\beta} u_0. \end{cases} \tag{3.4}$$

We prove that for every $n \in \mathbb{N}$, the function v_n belongs to the space $\widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$. Then we will use an induction argument on n.

For $n = 0$, by assumption, $u_0 \in \widetilde{B}^{-\alpha,\infty}_{L^{q,\infty}}(\mathbb{R}^3)$. By Lemma 3.8,

$$\begin{cases} \sup\limits_{t\in(0,T)} t^{1/p} \| v_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} < \sup\limits_{t>0} t^{1/p} \| e^{-t(-\Delta)^\beta} u_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} < \infty, \\ t^{1/p} \| v_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} = t^{1/p} \| e^{-t(-\Delta)^\beta} u_0(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \to 0 \quad (\text{as } t \to 0). \end{cases} \tag{3.5}$$

By Lemma 3.2, we have $v_0 \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$.

Next we assume that $v_{n-1} \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$. Let $\varepsilon > 0$, as $v_{n-1} \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$, there exist two functions $v^1_{n-1} \in L^\infty((0,T), L^{q,\infty}(\mathbb{R}^3))$ and $v^2_{n-1} \in L^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$ such that $\|v^2_{n-1}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant \varepsilon$ and $v_{n-1} = v^1_{n-1} + v^2_{n-1}$. We have

$$\begin{aligned} B(v_{n-1}, v_{n-1}) &= B\big(v^1_{n-1} + v^2_{n-1}, v^1_{n-1} + v^2_{n-1}\big) \\ &= B\big(v^1_{n-1}, v^1_{n-1}\big) + B\big(v^2_{n-1}, v^1_{n-1}\big) + B\big(v^1_{n-1}, v^2_{n-1}\big) + B\big(v^2_{n-1}, v^2_{n-1}\big) \\ &:= M_1 + M_2 \end{aligned}$$

with

$$M_1 = B\big(v^1_{n-1}, v^1_{n-1}\big) + B\big(v^2_{n-1}, v^1_{n-1}\big) + B\big(v^1_{n-1}, v^2_{n-1}\big) \quad \text{and} \quad M_2 = B\big(v^2_{n-1}, v^2_{n-1}\big).$$

By Lemma 2.1, we get

$$\begin{aligned} \|M_1\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} &\leqslant \big\| B\big(v^1_{n-1}, v^1_{n-1}\big)\big\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + \big\| B\big(v^1_{n-1}, v^2_{n-1}\big)\big\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \\ &\quad + \big\| B\big(v^2_{n-1}, v^1_{n-1}\big)\big\|_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \\ &\leqslant C \big\| v^1_{n-1}\big\|^2_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} + 2\big\| v^1_{n-1}\big\|^2_{L^\infty((0,T),L^{q,\infty}(\mathbb{R}^3))} \big\| v^2_{n-1}\big\|^2_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \\ &\leqslant C \end{aligned}$$

and $\|M_2\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \lesssim \|v^2_{n-1}\|^2_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \lesssim \varepsilon^2$. Thus, according to Proposition 1.7, we have $B(v_{n-1}, v_{n-1}) \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$.

We will prove that for every $n \in \mathbb{N}$,

$$\|v_n\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant 2 \big\| e^{-t(-\Delta)^\beta} u_0\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.$$

Since $v_0 = e^{-t(-\Delta)^\beta} u_0$, it is obvious that

$$\|v_0\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant 2 \big\| e^{-t(-\Delta)^\beta} u_0\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.$$

Assume that this is true for an $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|v_{n+1}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} &\leqslant \|v_0\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} + \big\| B(v_n, v_n)\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \\ &\leqslant \big\| e^{-t(-\Delta)^\beta} u_0\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} + 4C \big\| e^{-t(-\Delta)^\beta} u_0\big\|^2_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}. \end{aligned}$$

Taking $4C\|e^{-t(-\Delta)^\beta} u_0\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} < 1$, we get

$$\|v_{n+1}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \leqslant 2 \big\| e^{-t(-\Delta)^\beta} u_0\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))},$$

that is, $\|v_{n+1}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}$ in the ball centered at 0, of radius $2\|e^{-t(-\Delta)^\beta} u_0\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}$. Then,

$$\begin{aligned}
&\|v_n - v_{n-1}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \\
&\quad\leqslant \big\|B(v_{n-1} - v_{n-2}, v_{n-1})\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} + \big\|B(v_{n-2}, v_{n-1} - v_{n-2})\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} \\
&\quad\leqslant C\|v_{n-1} - v_{n-2}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\big(\|v_{n-1}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))} + \|v_{n-2}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\big) \\
&\quad\leqslant 4C\big\|e^{-t(-\Delta)^\beta}u_0\big\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}\|v_{n-1} - v_{n-2}\|_{L^{p,\infty}((0,T),L^{q,\infty}(\mathbb{R}^3))}.
\end{aligned}$$

Thus, the Picard contraction principle completes the proof. □

Now, we want to give the reverse result of Theorem 3.9. To do this, we need the following lemma.

Lemma 3.10. *Let $\frac{1}{2} < \beta < 1$ and let $\frac{3}{2\beta-1} < q < \infty$ and $\frac{2\beta}{2\beta-1} < p < \infty$ such that $\frac{2\beta}{(2\beta-1)p} + \frac{3}{(2\beta-1)q} = 1$ and $u \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$ be a mild solution to Eqs. (1.1). Then, for $0 < \varepsilon < 1$, there exists $0 < t_0 < T$ such that $\forall t \in (0, t_0]$, $\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{\varepsilon}{2C_0 t^{1/p}}$.*

Proof. It follows from $u \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$ that for all $\lambda > 0$, there exists a constant $C(\lambda)$, depending on λ, such that $C(\lambda) \to 0$ (as $\lambda \to \infty$) and

$$\big|\big\{\big\|u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)} > \lambda\big\}\big| < \frac{C(\lambda)}{\lambda^p}.$$

Let $0 < \varepsilon < 1$ and $0 < t_0 < 1$. Denote $\lambda_{t_0} = \frac{\varepsilon}{4C_0 t_0^{1/p}}$. When $t_0 \to 0$ and $C(\lambda_{t_0}) \to 0$, we choose t_0 small enough such that $C(\lambda_{t_0}) < \frac{\varepsilon^p}{2\times 4^{2p} C_0^p}$. Let $t \leqslant t_0$ such that $\lambda_t = \frac{\varepsilon}{4C_0 t^{1/p}} \geqslant \lambda_{t_0}$, then $C(\lambda_t) \leqslant C(\lambda_{t_0})$. We can get

$$\big|\big\{t \in (0,T),\ \big\|u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)} > \lambda_t\big\}\big| < \frac{C(\lambda_t)}{\lambda_t^p} < \frac{t}{2\times 4^p}. \tag{3.6}$$

We claim that there exists θ such that

$$t - \frac{t}{4^p} \leqslant \theta \leqslant t \quad \text{and} \quad \big\|u(\theta)\big\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{\varepsilon}{4C_0 t^{1/p}} = \lambda_t.$$

Otherwise

$$\big|\big\{t \in (0,T),\ \big\|u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)} > \lambda_t\big\}\big| \geqslant \left|\left[t - \frac{t}{4^p}, t\right]\right| = \frac{t}{4^p}.$$

This is a contraction to (3.6). Let $T^* = (4C_0\|u(\theta)\|_{L^{q,\infty}(\mathbb{R}^3)})^{-p}$. Taking $0 < \varepsilon < 1$, we have

$$\big\|u(\theta)\big\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t^{1/p}} \quad \Longrightarrow \quad t \leqslant \big(4C_0\big\|u(\theta)\big\|_{L^{q,\infty}(\mathbb{R}^3)}\big)^{-p}. \tag{3.7}$$

Applying Lemma 2.2 in the interval $[\theta, \theta + T^*]$, there exists a solution $\tilde{u} \in L^\infty((\theta, \theta + T^*), L^{q,\infty}(\mathbb{R}^3))$ to Eqs. (1.1). Note that (3.7) implies that

$$(\theta, t] \subset (\theta, \theta + t) \subset \big(\theta, \theta + T^*\big).$$

By Proposition 3.1, we know $u = \tilde{u}$ on $(\theta, t]$. So for $t \leqslant t_0$, there exists $0 < \theta < t$ such that $u \in L^\infty((\theta, t], L^{q,\infty}(\mathbb{R}^3))$ and

$$\forall s \in (\theta, t], \quad \big\|u(s)\big\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant 2\big\|u(\theta)\big\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{\varepsilon}{2C_0 t^{1/p}}.$$

This completes the proof of this lemma. □

Theorem 3.11. *For $\frac{1}{2} < \beta < 1$ and let $\frac{3}{2\beta-1} < q < \infty$ and $\frac{2\beta}{2\beta-1} < p < \infty$ such that $\frac{2\beta}{(2\beta-1)p} + \frac{3}{(2\beta-1)q} = 1$ and $u \in \widetilde{L}^{p,\infty}((0,T), L^{q,\infty}(\mathbb{R}^3))$ be a mild solution to Eqs. (1.1). Then*

$$\begin{cases} \sup\limits_{t\in(0,T)} t^{1/p}\big\|u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)} < \infty, \\ t^{1/p}\big\|u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)} \to 0 \quad (\textit{as } t \to 0). \end{cases} \tag{3.8}$$

Proof. By Lemma 3.10, for every $\varepsilon > 0$, there exists t_0 such that, for all $t \in (0, t_0)$,

$$t^{1/p}\big\|u(t)\big\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{\varepsilon}{2C_0},$$

that is, $\lim_{t\to 0} t^{1/p}\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} = 0$.

Now we prove the first assertion of (3.8). Checking the proof of Lemma 3.10 and taking $\varepsilon = \frac{1}{2}$, we can see that there exist t_0 such that for every $t \leqslant t_0$ and $0 < \theta < t$ such that $u \in L^\infty((\theta, t], L^{q,\infty}(\mathbb{R}^3))$ and

$$\forall s \in (\theta, t], \quad \|u(s)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant 2\|u(\theta)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t^{1/p}}. \tag{3.9}$$

On the other hand, Lemma 3.10 and $\lim_{t\to 0} t^{1/p}\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} = 0$ tell us that there exists t_1 such that for $s \in (0, t_1)$, $t^{1/p}\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant C$. If $t_0 > t_1$, take $t_2 < t_1 < t_0$ (otherwise take $t_2 = t_0$). By (3.9), there exists θ_2 such that for every $s \in (\theta_2, t_2]$, $\|u(s)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t_2^{1/p}}$. Because $t^{1/p}\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)}$ is bounded on $(0, \theta_2] \subset (0, t_1)$, now we restrict $t \in (\theta_2, T]$. Define a new function $\tilde{u}(s) = u(t_2 - \theta_2 + s)$. Then we only need to prove the assertion for $\tilde{u}(s)$ on $s \in (\theta_2, T + t_2 - \theta_2]$.

Since u is a solution to Eqs. (1.1),

$$u \in \widetilde{L}^{p,\infty}\big((t_2, T), L^{q,\infty}(\mathbb{R}^3)\big) \quad \Longrightarrow \quad \tilde{u} \in \widetilde{L}^{p,\infty}\big((\theta_2, T - t_0 + \theta_2), L^{q,\infty}(\mathbb{R}^3)\big)$$

implies that $\tilde{u}$ is also a solution to Eqs. (1.1). By Lemma 3.10 for $\varepsilon = \frac{1}{2}$ again, we can get that for $\forall t \in (\theta_2, t_2)$, $\|\tilde{u}(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t_2^{1/p}}$. That is, $\forall t \in (t_2, 2t_2 - \theta_2)$, $\|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t_2^{1/p}}$. We conclude that

$$\forall t \in (\theta_2, 2t_2 - \theta_2), \quad \|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t_2^{1/p}}.$$

Since T is finite, we can find a constant $n \in \mathbb{N}$ such that $nt_2 < T < (n+1)t_2$. Hence repeating this argument finite many times, we get

$$\forall t \in (\theta_2, T], \quad \|u(t)\|_{L^{q,\infty}(\mathbb{R}^3)} \leqslant \frac{1}{4C_0 t_2^{1/p}} < \frac{1}{4C_0 t_2^{1/p}} \frac{T^{1/p}}{t^{1/p}}.$$

This completes the proof of this theorem. $\square$

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